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MATHEMATICS

ALGEBRAS OF BOUNDED REAL-VALUED FUNCTIONS. I

BY

R. B. KIRK

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Introduction

Let X be a set and let A be a uniformly closed algebra of bounded real-valued functions on X. Also assume that A contains the constants and separates the points of X. The object of this paper will be to study such algebras when they satisfy a condition called z-separating. A typical z-separating algebra is the space $C^b(X)$ of all bounded, continuous real-valued functions on a completely regular Hausdorff space X. And in fact, it will be seen that z-separating algebras possess many properties in common with such continuous function spaces.

In section 1 the structure of these algebras is discussed and some preliminary results are obtained. In sections 2 and 3, z-separating algebras are discussed at length. Several characterizations of such algebras are given in proposition 2.5. (Condition 5 of this proposition is new even for the continuous function case.) Also a number of interesting examples of z-separating algebras are described. Finally a new characterization of pseudocompact spaces is given in terms of z-separating algebras (theorem 3.9).

In section 4 is contained a study of spaces of regular finitely-additive set functions (which are used later to represent the dual spaces of z-separating algebras). In section 5 the representation of the Banach dual of a z-separating algebra is carried out. The result is theorem 5.11 which contains the representation theorems of $C^b(X)^*$ for normal spaces ([2], p. 262) and $C^b(X)^*$ for completely regular Hausdorff spaces ([1]) as special cases. It also yields representations for the dual spaces of the z-separating algebras in examples 3.3, 3.4 and 3.5. Also of interest is proposition 5.4 which shows that the z-separating algebras are precisely the algebras for which a representation of the dual may be given in the "usual" way as a space of regular finitely-additive set functions.

Finally in section 6, it is shown how the set functions in the dual of a z-separating algebra are related to the regular Borel measures on a compact Hausdorff space X_A associated with A. In particular, if X is a completely regular Hausdorff space, the theorem describes how regular finitely-additive set functions on X may be obtained from the regular Borel measures on βX (the Stone-Cech compactification of X).

1. *Preliminary Remarks*

As above let A be an algebra of bounded real-valued functions on X which contains the constants and separates the points of X. Also assume that A is complete with respect to the uniform norm. Hence A is a real Banach algebra. Furthermore, with respect to the usual pointwise ordering, A is an M-space with a unit in the sense of Kakutani [7]. Thus there is a compact Hausdorff space X_A such that A is isomorphic as a Banach lattice to $C(X_A)$.

We briefly indicate how the space X_A may be realized. Let Y denote the set of all non-zero algebra homomorphisms of A onto R. (It is not hard to show that every algebra homomorphism of A into R is also continuous and lattice-preserving.) Then Y is a subset of the unit ball in A^* (the Banach space dual of A). Furthermore, Y is closed in A^* with respect to the weak*-topology and hence compact by the Alaoglu theorem. Define X_A to by Y with this topology. The isomorphism $I: A \to C(X_A)$ is then defined by $If(\phi) = \phi(f)$ for all $\phi \in X_A$.

The set X may be imbedded in X_A as follows. For each $x \in X$, let $\phi_x(f) = f(x)$ for all $f \in A$. It is clear that $\phi_x \in X_A$. Furthermore, since A separates the points of X, the map $x \to \phi_x$ is an injection of X in X_A. From now on we will identify X with its image under this injection and treat X as a subset of X_A.

As a subset of X_A, X inherits a completely regular Hausdorff topology which we denote by τ_A. It is not hard to verify that τ_A is the weakest topology on X for which the functions in A are continuous. Thus A is a closed subalgebra of the algebra $C^b(X, \tau_A)$ of all bounded, real-valued τ_A-continuous functions on X. However, in general, $A \neq C^b(X, \tau_A)$.

Proposition 1.1 *(X, τ_A) is dense in X_A.*

Proof. Let $\phi_0 \in X_A$ and assume that ϕ_0 does not belong to the closure of X in X_A. Then there is an $\varepsilon > 0$ and functions $f_1, \ldots, f_n \in A$ such that

$$X \cap \{\phi \in X_A : |\phi(f_i) - \phi_0(f_i)| < \varepsilon \text{ for } i = 1, \ldots, n\} = \emptyset.$$

Define $f = \sum_1^n |f_i - \phi_0(f_i)|$. Then since A is a lattice, $f \in A$, and $f^{-1} \in A$ since $f(x) \geqslant \varepsilon$ for all $x \in X$. Also $\phi_0(f) = 0$ since ϕ_0 is lattice preserving. But then $\phi_0(1) = \phi_0(f) \cdot \phi_0(f^{-1}) = 0$ which contradicts the fact that $\phi_0 \neq 0$.

From the above proposition, it follows that X_A is a compactification of (X, τ_A). Also it is not difficult to show that A is exactly the set of functions in $C^b(X, \tau_A)$ which can be extended uniquely as continuous functions to X_A. Thus it is clear that X_A is the Smirnov compactification for the totally-bounded uniformity on X determined by the algebra A. (Of course, A is just the algebra of bounded uniformly continuous functions on X for this uniformity.)

We will now discuss briefly how the topology on X_A is determined intrinsically by that of X. Much of what follows is analogous to the relation between X and βX as discussed in ([3]).

Definition 1.2 For each $f \in A$, let $Z(f)=\{x \in X: f(x)=0\}$. A subset $Z \subset X$ is an *A-zero* set if $Z=Z(f)$ for some $f \in A$. Denote by $\mathscr{Z}(A)$ the set of all A-zero sets of X.

The family $\mathscr{Z}(A)$ forms a basis for the closed sets in the topology τ_A. The following properties are easily verified.

Proposition 1.3

a) $X, \phi \in \mathscr{Z}(A)$.

b) *If* $Z_1, Z_2 \in \mathscr{Z}(A)$, *then* $Z_1 \cup Z_2 \in \mathscr{Z}(A)$.

c) *If* $\{Z_n: n \in N\} \subset \mathscr{Z}(A)$, *then* $\cap \{Z_n: n \in N\} \in \mathscr{Z}(A)$.

We now prove some preliminary results. If $f \in A$, we will let $\hat{f}$ denote the unique extension of f to X_A. Also if Y is a subset of X, $\overline{Y}$ will always denote the closure of Y in X_A. (We will maintain this convention throughout the paper.)

Lemma 1.4 *Let* $0 \leqslant f \in A$ *and let* $Z=Z(\hat{f})$. *If* $\varepsilon > 0$ *and if* $Z_\varepsilon = Z((f-\varepsilon)^+)$, *then* $Z \subset \overline{Z}_\varepsilon \subset Z((\hat{f}-\varepsilon)^+)$.

Proof. It is immediate that $\overline{Z}_\varepsilon \subset Z((\hat{f}-\varepsilon)^+)$. Let $x \in Z$ and let $\{x_i\}$ be a net in X such that $x_i \to x$. (X is dense in X_A). Then $\hat{f}(x_i)=f(x_i) \to \to \hat{f}(x)=0$. Thus $\{x_i\}$ is eventually in Z_ε which implies that $x \in \overline{Z}_\varepsilon$.

Proposition 1.5 *If* $Z \in \mathscr{Z}(C(X_A))$, *then there is a set* $\{Z_n: n \in N\} \subset \subset \mathscr{Z}(A)$ *such that* $Z = \cap \{\overline{Z}_n: n \in N\}$.

Proof. Let $Z=Z(\hat{f})$ where $f \in A$. For each $n \in N$, it follows from lemma 1.4 that $Z \subset \overline{Z}_n \subset Z((|\hat{f}|-n^{-1})^+)$, where $Z_n = Z((|f|-n^{-1})^+)$. Hence we have that

$$Z \subset \cap \{\overline{Z}_n: n \in N\} \subset \cap \{Z((|\hat{f}|-n^{-1})^+): n \in N\} = Z.$$

2. *Separating Algebras.*

Definition 2.1 In the following, $\mathscr{U}(A)$ will denote a family of closed subsets of X with the following properties.

1. $\mathscr{U}(A)$ is a base for the τ_A-closed sets in X.
2. If $G_1, G_2 \in \mathscr{U}(A)$, then $G_1 \cap G_2$, $G_1 \cup G_2 \in \mathscr{U}(A)$.
3. $\mathscr{Z}(A) \subset \mathscr{U}(A)$.

Of course, $\mathscr{Z}(A)$ satisfies the above conditions as does the family of all closed subsets of X.

Definition 2.2 The algebra A *separates* $\mathscr{U}(A)$ if the following is satisfied. If $G_1, G_2 \in \mathscr{U}(A)$ and if $G_1 \cap G_2 = \emptyset$, then there is a function $f \in A$ such that $f(x)=0$ whenever $x \in G_1$ and $f(x)=1$ whenever $x \in G_2$. If A separates $\mathscr{Z}(A)$, we will say that A is *z-separating*.

It is clear that if A separates $\mathscr{U}(A)$, then the function f in the above definition may be taken such that $0 \leqslant f \leqslant 1$. Define a family of closed subsets of X_A by $\overline{\mathscr{U}}(A)=\{\overline{G}: G \in \mathscr{U}(A)\}$. Since $\mathscr{Z}(A) \subset \mathscr{U}(A)$, it follows from proposition 1.5 that $\overline{\mathscr{U}}(A)$ is a base for the closed sets in X_A. Let $\mathscr{F}(\mathscr{U}(A))$ denote the Boolean algebra of subsets of X generated by $\mathscr{U}(A)$ and let $\mathscr{F}(\overline{\mathscr{U}}(A))$ denote the algebra of subsets of X_A generated by $\overline{\mathscr{U}}(A)$. The following is easily proved.

Proposition 2.3 *Let F be a subset of X. Then $F \in \mathscr{F}(\mathscr{U}(A))$ if and only if there are sets $G_1, \ldots, G_n$ and $H_1, \ldots, H_n$ in $\mathscr{U}(A)$ such that the following conditions hold.*

1. *For $i=1, \ldots, n$, $H_i \subset G_i$.*
2. *For $i \neq j$, $(G_i - H_i) \cap (G_j - H_j) = \emptyset$.*
3. *$F = \cup \{G_i - H_i : i=1, \ldots, n\}$.*

We will need the following result due to Sikorski ([8], p. 37).

Proposition 2.4 *Let $\mathscr{B}_1$ and $\mathscr{B}_2$ be Boolean algebras and assume that the sets $\mathscr{G}_1$ and $\mathscr{G}_2$ generate $\mathscr{B}_1$ and $\mathscr{B}_2$ respectively. Furthermore, let σ by a one-one mapping of $\mathscr{G}_1$ onto $\mathscr{G}_2$. Then σ can be extended to a Boolean isomorphism of $\mathscr{B}_1$ onto $\mathscr{B}_2$ if and only if for every sequence $a_1, \ldots, a_n \in \mathscr{G}_1$ and for every sequence of numbers $\varepsilon_1, \ldots, \varepsilon_n$ (where ε_i is either 1 or -1 for $i=1, \ldots, n$), the following holds.*

(*) $\quad \varepsilon_1 a_1 \wedge \ldots \wedge \varepsilon_n a_n = 0$ *if and only if* $\varepsilon_1 \sigma(a_1) \wedge \ldots \wedge \varepsilon_n \sigma(a_n) = 0.$

(Here, of course, $1x$ and $-1x$ denote x and the complement of x respectively for every x in the Boolean algebra.)

Proposition 2.5 *Let $\mathscr{U}(A)$ satisfy the conditions of definition* 2.1. *Then the following statements are equivalent.*

1. *A separates $\mathscr{U}(A)$.*
2. *Let $G_1, G_2 \in \mathscr{U}(A)$. If $G_1 \cap G_2 = \emptyset$, then $\bar{G}_1 \cap \bar{G}_2 = \emptyset$.*
3. *Let $G_1, G_2 \in \mathscr{U}(A)$. Then $\overline{G_1 \cap G_2} = \bar{G}_1 \cap \bar{G}_2$.*
4. *Let $G_1, G_2, H_1, H_2 \in \mathscr{U}(A)$. If $(G_1 - H_1) \cap (G_2 - H_2) = \emptyset$, then $(\bar{G}_1 - \bar{H}_1) \cap (\bar{G} - \bar{H}_2) = \emptyset$.*
5. *The mapping $\sigma : \mathscr{U}(A) \to \overline{\mathscr{U}}(A)$ defined by $\sigma(G) = \bar{G}$ can be extended to a Boolean algebra isomorphism of $\mathscr{F}(\mathscr{U}(A))$ onto $\mathscr{F}(\overline{\mathscr{U}}(A))$.*

Proof. $(1 \Rightarrow 2)$ Let $G_1, G_2 \in \mathscr{U}(A)$ be such that $G_1 \cap G_2 = \emptyset$. Then there is a function $f \in A$ such that $0 \leqslant f \leqslant 1$, $f(x) = 0$ whenever $x \in G_1$ and $f(x) = 1$ whenever $x \in G_2$. Thus we have that $\bar{G}_1 \subset Z(\hat{f})$ and $\bar{G}_2 \subset Z(1 - \hat{f})$. Hence $\bar{G}_1 \cap \bar{G}_2 = \emptyset$.

$(2 \Rightarrow 3)$ We must show that $\bar{G}_1 \cap \bar{G}_2 \subset \overline{G_1 \cap G_2}$. Hence let $x_0 \in \bar{G}_1 \cap \bar{G}_2$. Let W be any zero set neighborhood of x_0 in X_A. (That is, $W = \{x \in X_A : \hat{f}(x) > 0\}$ for some $f \in A$ with $\hat{f}(x_0) > 0$.) Since the zero set neighborhoods in X_A form a base for the open sets, it is sufficient to show that $W \cap (G_1 \cap G_2) \neq \emptyset$.

Set $U = W \cap X = \{x \in X : f(x) > 0\}$. Then $x_0 \in \overline{U \cap G_1}$. Indeed, if V is any zero set neighborhood of x_0 in X_A, then

$$V \cap (U \cap G_1) = (V \cap W) \cap G_1 \neq \emptyset,$$

since $x_0 \in \bar{G}_1$. Similarly it follows that $x_0 \in \overline{U \cap G_2}$. Thus, $(\overline{U \cap G_1})$ $(\overline{U \cap G_2}) \neq \emptyset$. Since U is an A-zero set $U \cap G_1$, $U \cap G_2 \in \mathscr{U}(A)$. Hence by condition 2, $U \cap (G_1 \cap G_2) \neq \emptyset$. Thus $x_0 \in \overline{G_1 \cap G_2}$.

$(3 \Rightarrow 2)$ This is obvious.

$(2 \Rightarrow 1)$ Assume that $G_1 \cap G_2 = \emptyset$. Then by assumption, $G_1 \cap G_2 = \emptyset$. Since X_A is a compact Hausdorff space, by Urysohn's lemma, there is $g \in C(X_A)$ such that $g(x) = 0$ for all $x \in \bar{G}_1$ and $g(x) = 1$ for all $x \in \bar{G}_2$. Since the mapping $I: A \to C(X_A)$ is onto, there is an $f \in A$ such that $g = \hat{f}$. Then $f(x) = 0$ for all $x \in G_1$ and $f(x) = 1$ for all $x \in G_2$. Thus A separates $\mathscr{U}(A)$.

$(3 \Rightarrow 4)$ Let $x_0 \in (\bar{G}_1 - \bar{H}_1) \cap (\bar{G}_2 - \bar{H}_2)$. By condition 3,

$$(\bar{G}_1 - \bar{H}_1) \cap (\bar{G}_2 - \bar{H}_2) = (\overline{G_1 \cap G_2}) - (\overline{H_1 \cup H_2}).$$

Hence $x_0 \in \overline{G_1 \cap G_1}$ and $x_0 \notin \overline{H_1 \cup H_2}$. Let U be a zero set neighborhood of x_0 in X_A such that $U \cap (H_1 \cup H_2) = \emptyset$ and let $y_0 \in U \cap (G_1 \cap G_2)$. Then $y_0 \in (G_1 - H_1) \cap (G_2 - H_2)$.

$(4 \Rightarrow 2)$ The result follows immediately by taking $H_1 = H_2 = \emptyset$.

$(4 \Rightarrow 5)$ Note first that the mapping σ is one-one. (Indeed, if $\bar{G}_1 = \bar{G}_2$, then $G_1 = X \cap \bar{G}_1 = X\ \bar{G}_2 = G_2$ since G_1 and G_2 are closed in X.) We will now show that condition (*) of proposition 2.4 is satisfied. Indeed, if $G_1, \ldots, G_n, H_1, \ldots, H_n$ are elements of $\mathscr{U}(A)$, set

$$G = \cap \{G_i : i = 1, \ldots, n\} \text{ and } H = \cup \{H_i : i = 1, \ldots, n\}.$$

If

$$G_1 \cap \ldots \cap G_n \cap (H_1)^c \cap \ldots \cap (H_n)^c = G - H = \emptyset,$$

then $(G - H) \cap (X - \emptyset) = \emptyset$. Since $G, H, X, \emptyset \in \mathscr{U}(A)$, condition 4 implies that $(\bar{G} - \bar{H}) \cap (X_A - \emptyset) = \theta$. That is, $\bar{G} - \bar{H} = \emptyset$. From condition 3 (which we have already seen is implied by condition 4), it follows that $\bar{G} = \cap \{\bar{G}_i : i = 1, \ldots, n\}$. Thus it follows that $\bar{G}_1 \cap \ldots \cap \bar{G}_n \cap (\bar{H}_1)^c \cap \ldots \cap (\bar{H}_n)^c = \bar{G} - \bar{H} = \emptyset$. On the other hand, if $\bar{G}_1 \cap \ldots \cap \bar{G}_n \cap (\bar{H}_1)^c \cap \ldots \cap (\bar{H}_n)^c = \emptyset$, then it follows that

$$G_1 \cap \ldots \cap G_n \cap (H_1)^c \cap \ldots \cap (H_n)^c = X \cap [\bar{G}_1 \cap \ldots \cap \bar{G}_n \cap (\bar{H}_1)^c \cap \ldots \cap (\bar{H}_n)^c] = \emptyset.$$

Hence by proposition 2.4, σ may be extended to an isomorphism of $\mathscr{F}(\mathscr{U}(A))$ onto $\mathscr{F}(\bar{\mathscr{U}}(A))$.

$(5 \Rightarrow 2)$ As above it follows that σ is one-one. Hence by proposition 2.4, if $G_1, G_2 \in \mathscr{U}(A)$ and if $G_1 \cap G_2 = \emptyset$, then $\bar{G}_1 \cap \bar{G}_2 = \sigma(G_1) \cap \sigma(G_2) = \emptyset$. The proof is complete.

The following is an immediate consequence of proposition 2.3 and condition 5 of the above proposition.

Proposition 2.6 *Let A separate $\mathscr{U}(A)$ and let F be a subset of X_A. Then $F \in \mathscr{F}(\overline{\mathscr{U}}(A))$ if and only if there are sets $G_1, \ldots, G_n$ and $H_1, \ldots, H_n$ in $\mathscr{U}(A)$ such that the following conditions hold.*

1. *For $i=1, \ldots, n$, $\bar{H}_i \subset \bar{G}_i$.*
2. *For $i \neq j$, $(\bar{G}_i - \bar{H}_i) \cap (\bar{G}_j - \bar{H}_j) = \emptyset$.*
3. *$F = \cup \{\bar{G}_i - \bar{H}_i : i=1, \ldots, n\}$.*

3. *Z-separating Algebras*

In this section we will consider questions about z-separating algebras. We begin with some examples. Let (X, τ) be a completely regular Hausdorff space. (In general, τ will not be the topology τ_A on X determined by the algebras considered below.) Let $\mathscr{A}$ be a family of subsets of X such that 1) $\emptyset \in \mathscr{A}$ and 2) $\cup \{X_n : n \in N\} \in \mathscr{A}$ whenever $\{X_n : n \in N\} \subset \mathscr{A}$. Let $A_{\mathscr{A}}$ be the set of all bounded, real-valued functions f on X such that f is continuous on $X-Y$ for some $Y \in \mathscr{A}$. (That is, the restriction of f to $X-Y$ is continuous with respect to the relative topology on $X-Y$.) Note that if f is a τ-continuous bounded real-valued function on X, then $f \in A_{\mathscr{A}}$ since $\emptyset \in \mathscr{A}$.

Proposition 3.1 *The set $A_{\mathscr{A}}$ is a z-separating algebra.*

Proof. It is easy to verify that $A_{\mathscr{A}}$ is an algebra of bounded, real-valued functions on X which contains the constants and separates the points of X. In order to see that $A_{\mathscr{A}}$ is uniformly closed, let $\{f_n\} \subset A_{\mathscr{A}}$ and assume that $f_n \to f$ uniformly on X. Take $X_n \in \mathscr{A}$ such that f_n is continuous on $X-X_n$. Then f is continuous on $X - \cup \{X_n : n \in N\}$. Thus $f \in A_{\mathscr{A}}$.

Now take $Z_1, Z_2 \in \mathscr{Z}(A)$ such that $Z_1 \cap Z_2 = \emptyset$. Take $f, f_2 \in A$ such that $Z_1 = Z(f_1)$ and $Z_2 = Z(f_2)$. Now define $f = f_1^2 \cdot (f_1^2 + f_2^2)^{-1}$. It is clear that f is bounded. (In fact, $0 \leqslant f \leqslant 1$.) Furthermore, if f_1 is continuous on $X - X_1$ and f_2 is continuous on $X-X_2$ with $X_1, X_2 \in \mathscr{A}$, then f is continuous on $X-(X_1 \cup X_2)$. Hence $f \in A_{\mathscr{A}}$. Since $f(x)=0$ for $x \in Z_1$ and $f(x)=1$ for $x \in Z_2$, it follows that $A_{\mathscr{A}}$ is z-separating.

We now list some particular cases of the family of z-separating algebras described above.

Example 3.2 If $\mathscr{A} = \{\emptyset\}$, then $A_{\mathscr{A}} = C^b(X, \tau)$.

Example 3.3 Let A be the set of all bounded real-valued functions on (X, τ) which are continuous except on a set of the first category. Then A is a z-separating algebra. (Take $\mathscr{A}$ to be the family of subsets of (X, τ) of the first category.)

Example 3.4 Let m be a measure (defined for some σ-algebra of subsets of X.) Let A be the set of bounded real-valued functions on X which are continuous except on a set of m-measure zero. Then A is a z-separating algebra.

Example 3.5 Let I be any interval (finite or infinite) in the set R of real numbers. Then the set A of all bounded locally-Riemann integrable functions on I is a z-separating algebra. (Indeed, $A = A_{\mathscr{A}}$ where $\mathscr{A}$ is the family of all subsets of I of Lebesgue measure zero.)

Lest the reader be tempted to conjecture that all algebras are z-separating, the next theorem will show in effect that there are many algebras A which are not z-separating. Before stating it, however, some preliminary discussion is in order. If (X, τ) is a completely regular Hausdorff space, a uniformly closed subalgebra of $C^b(x, \tau)$ which contains the constants and separates the points of X will be called an *SW-algebra*. (The SW stands for Stone-Wierstrass.) Following Gillman-Jerrison [3], let βX and νX denote the Stone-Cech compactification and the realcompactification of (X, τ) respectively. Finally the recall that (X, τ) is called pseudocompact if every τ-continuous real-valued function on X is bounded.

Lemma 3.6 *Let (X, τ) be a completely regular Hausdorff space. Assume that $x_1, x_2, \in \beta X - \nu X$ with $x_1 \neq x_2$. Then there are functions $g_1, g_2 \in$ $\in C^b(X, \tau)$ such that the following conditions hold.*

1. $Z(g_1) \cap Z(g_2) = \emptyset$.
2. $x_1, x_2 \in Z(\hat{g}_1) \cap Z(\hat{g}_2)$ *where $\hat{g}_i$ is the unique extension of g_i to βX for $i = 1, 2$.*
3. *For $i = 1, 2$, $x_i \in cl_{\beta X}\ (Z(gi))$.*

Proof. Since $x_1, x_2 \in \beta X - \nu X$, there are zero sets $Z_1, Z_2 \in (C(\beta X))$ such that $x_1 \in Z_1$, $x_2 \in Z_2$, $Z_1 \cap Z_2 = \emptyset$ and $X \cap Z_1 = X \cap Z_2 = \emptyset$. Let $f \in C^b(X)$ be such that $Z_1 = Z(\hat{f})$ and $Z_2 = Z(1 - \hat{f})$. Define sets

$$W_1 = \{x \in \beta X : \hat{f}(x) \leqslant 1/3\} \text{ and } W_2 = \{x \in \beta X : \hat{f}(x) \geqslant 2/3\}.$$

Let $g_1, g_2 \in C^b(X)$ be such that $Z(\hat{g}_1) = W_1 \cup Z_2$ and $Z(\hat{g}_2) = W_2 \cup Z_1$. Then

$$Z(g_1) \cap Z(g_2) = X \cap (W_1 \cup Z_2) \cap (W_2 \cup Z_1) = X \cap W_1 \cap W_2 = \emptyset,$$

and $x_1, x_2 \in Z(g_1) \cap Z(g_2)$. Finally for $i = 1, 2$, $x_i \in cl_{\beta X}\ (Z(g_i))$. Indeed, if W is any zero set neighborhood of x_i in βX, then $W \cap W_i$ is a zero set neighborhood of x_i and so $X \cap W \cap W_i \neq \emptyset$. But $W \cap Z(g_i) = X \cap W \cap W_i$. The proof is complete.

The following is from [3], p. 136.

Lemma 3.7 *Let (X, τ) be a completely regular Hausdorff space. Then $\beta X - \nu X$ is either empty or it has cardinality at least 2^c.*

Lemma 3.8 *Let (X, τ) be a completely regular Hausdorff space and let $x_1, x_2 \in \beta X - \nu X$, with $x_1 \neq x_2$. Then the set $A = \{f \in C^b(X) : \hat{f}(x_1) = \hat{f}(x_2)\}$ is an SW-algebra which is not z-separating.*

Proof. It is an easy matter to verify that A is an SW-algebra. In order to see that A is not z-separating, let $g_1, g_2 \in C^b(X)$ be taken as in lemma 3.6. Then $g_1, g_2 \in A$ by condition 2 of that lemma. Also we have that $Z(g_1) \cap Z(g_2) = \emptyset$. Now if $f \in C^b(X)$ is such that $f(x) = 0$ for all $x \in Z(g_1)$ and $f(x) = 1$ for all $x \in Z(g_2)$, then $\hat{f}(x_1) = 0$ and $\hat{f}(x_2) = 1$ by condition 3 of lemma 3.6. Thus $f \notin A$ and so A is not z-separating.

Theorem 3.9 *Let (X, τ) be a completely regular Hausdorff space. The every SW-algebra on (X, τ) is z-separating if and only if (X, τ) is pseudocompact.*

Proof. ($\Leftarrow$) Let A be an SW-algebra which is not z-separating and take $Z_1, Z_2 \in \mathscr{Z}(A)$ such that $Z_1 \cap Z_2 = \emptyset$ but $\bar{Z}_1 \cap \bar{Z}_2 \neq \emptyset$. Take $f_1, f_2 \in A$ such that $Z_1 = Z(f_1)$ and $Z_2 = Z(f_2)$. Set $g = (f_1^2 + f_2^2)^{-1}$. Then g is τ-continuous and unbounded. Thus (X, τ) is not pseudocompact.

($\Rightarrow$) Assume that X is not pseudocompact. This is equivalent to assuming that $\beta X - \nu X \neq \emptyset$. Then by lemma 3.7, there are two distinct points $x_1, x_2 \in \beta X - \nu X$. Hence by lemma 3.8, there is an SW-algebra on (X, τ) which is not z-separating.

From the above theorem, it follows that there are many examples of algebras which are not z-separating. However, we will now give one particular example which we will refer to occasionally below.

Example 3.10 Let X be the set of rational numbers in $[0, 1]$ with its usual topology. Let A be the set of functions on X which are uniformly continuous on X. (That is, A is the set of restrictions to X of the functions continuous on $[0, 1]$.) It is obvious that A is a uniformly closed algebra of bounded continuous real-valued functions which contains the constants and separates the points of X. Furthermore, it is easy to see that $X_A = [0, 1]$.

In order to see that A is not z-separating, let α be any irrational number in $[0, 1]$. The set $Z_1 = \{x \in X : x < \alpha\}$ and $Z_2 = \{x \in X : \alpha < x\}$. Then $Z_1, Z_2 \in \mathscr{Z}(A)$ and $Z_1 \cap Z_2 = \emptyset$. However, $\bar{Z}_1 \cap \bar{Z}_2 = \{\alpha\}$. Hence A is not z-separating by condition 2 of proposition 2.5.

We will now briefly consider the following question. Let (X, τ) be a completely regular Hausdorff space and let A be an SW-algebra on X. When is $A = C^b(X, \tau)$? The Stone-Wierstrass theorem states that if (X, τ) is compact, then $C^b(X)$ is the only SW-algebra on (X, τ). It is interesting to note that this fact characterizes compact spaces as the following shows. (Compare [4], theorem 3.)

Proposition 3.11 *Let (X, τ) be a completely regular Hausdorff space. Then the following are equivalent.*

1. *(X, τ) is compact.*
2. *$C^b(X)$ is the only SW-algebra on X.*
3. *$C^b(X)$ is the only z-separating SW-algebra on X.*
4. *$C^b(X)$ is a minimal SW-algebra on (X, δ) where δ denotes the discrete topology on X.*

Proof. $(1 \Rightarrow 2)$ This is the Stone-Wierstrass theorem.

$(2 \Rightarrow 3)$ This is obvious.

$(3 \Rightarrow 1)$ If X is not compact, let $x_1 \in X$ and $x_2 \in \beta X - X$. Define $A = \{f \in C^b(X) \colon f(x_1) = \hat{f}(x_2)\}$. Then A is easily seen to be an SW-algebra on X different from $C^b(X)$. If $Z_1, Z_2 \in \mathscr{Z}(A)$, let $f_1, f_2 \in A$ be such that $Z_1 = Z(f_1)$ and $Z_2 = Z(f_2)$. If $Z_1 \cap Z_2 = \emptyset$, then $g = f_1^2 \cdot (f_1^2 + f_2^2)^{-1} \in C^b(X)$. Furthermore, $g(x_1) = \hat{g}(x_2)$ so that $g \in A$. Since $g(x) = 0$ for all $x \in Z_1$ and $g(x) = 1$ for all $x \in Z_2$, A is z-separating.

$(1 \Rightarrow 4)$ This is immediate from the Stone-Wierstrass theorem.

$(4 \Rightarrow 2)$ This is obvious.

Theorem 3.12 *Let (X, τ) be a completely regular Hausdorff space and let A be an SW-algebra on X. Then $A = C^b(X)$ if and only if the following two conditions hold.*

1. *A is z-separating.*
2. *$\mathscr{Z}(A) = \mathscr{Z}(C^b(X))$.*

Proof. $(\Rightarrow)$ This is obvious.

$(\Leftarrow)$ Let $\hat{A} = \{\hat{f} \colon f \in A\}$ denote the extensions of the elements of A to βX. If we can show that $\hat{A}$ separates the points of βX, it will follow from the Stone-Wierstrass theorem that $\hat{A} = C(\beta X)$ and hence that $A = C^b(X)$.

Hence take $x_1, x_2 \in \beta X$ with $x_1 \neq x_2$. Then there are zero set neighborhoods W_1 and W_2 of x_1 and x_2 respectively in βX such that $W_1 \cap W_2 = \emptyset$. Let $Z_1 = X \cap W_1$ and $Z_2 = X \cap W_2$. By condition 2, $Z_1, Z_2 \in \mathscr{Z}(A)$. By condition 1 there is an $f \in A$ such that $f(x) = 0$ for all $x \in Z_1$ and $f(x) = 1$ for all $x \in Z_2$. Hence $f(x_1) \neq f(x_2)$ and the proof is complete.

It is not difficult to see that neither condition in the theorem can be omitted. Indeed, example 3.5 is an SW-algebra on the space R with the discrete topology (which in fact is the topology τ_A for this example). On the other hand, the algebra A of example 3.10 satisfies condition 2 (since every closed set is an A-zero set) but fails to be z-separating.

Corollary 3.13 Let A be an SW-algebra on the completely regular space (X, τ). Then $A = C^b(X)$ if and only if A separates $(C^b(X))$.

(To be continued)